

# LATTICE DIAGRAM POLYNOMIALS IN ONE SET OF VARIABLES

J.-C. AVAL, F. BERGERON AND N. BERGERON

ABSTRACT. The space  $M_{\mu/i,j}$  spanned by all partial derivatives of the lattice polynomial  $\Delta_{\mu/i,j}(X; Y)$  is investigated in [5] and many conjectures are given. Here, we prove all these conjectures for the  $Y$ -free component  $M_{\mu/i,j}^0$  of  $M_{\mu/i,j}$ . In particular, we give an explicit bases for  $M_{\mu/i,j}^0$  which allow us to prove directly the central *four term recurrence* for these spaces.

**This paper is dedicated to the memory of Rodica Simion**

## 1. INTRODUCTION

We are going to explicitly describe certain  $\mathcal{S}_n$ -modules of polynomials, in  $n$  variables  $x_1, \dots, x_n$ , that are closely related to classical harmonic polynomials for the symmetric group  $\mathcal{S}_n$ . These last polynomials can be characterized by the fact that they satisfy the conditions

$$(1.1) \quad \sum_{i=1}^n \partial_{x_i}^k P(x_1, \dots, x_n) = 0, \quad k = 1, 2, 3, \dots$$

A classical result of Steinberg states that the set  $M_n$ , of all harmonic polynomials for the symmetric group, is

$$M_n := \mathcal{L}_\partial[\Delta_n],$$

where  $\Delta_n = \prod_{i < j} (x_i - x_j)$  is the Vandermonde determinant, and  $\mathcal{L}_\partial[\Delta_n]$  denotes the linear span of all partial derivatives of  $\Delta_n$ . It is certainly striking to notice that the dimension of  $M_n$  is  $n!$ , and there is a lot of other nice results related to  $M_n$  and its generalization to reflection groups. The spaces studied here are natural generalizations of these spaces and spaces studied by DeConcini and Procesi in [8] and Garsia and Procesi [11].

The point of departure of this work consists in replacing  $\Delta_n$  by natural generalizations of the Vandermonde determinant. To this end, let us define a general *lattice diagram* to be any finite subset of  $\mathbb{N} \times \mathbb{N}$ . The case corresponding to diagrams of partitions is of special interest. Recall that a *partition*  $\mu$  of  $n$  is a sequence

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$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$  of decreasing positive integers such that  $n = \mu_1 + \cdots + \mu_k$ . We denote  $\mu \vdash n$  this fact. For  $\mu \vdash n$ , we set

$$\mu! := \mu_1! \mu_2! \cdots \mu_k!.$$

The lattice diagram associated to a partition  $\mu$  is defined to be the set

$$\{(i, j) : 0 \leq i \leq k-1, 0 \leq j \leq \mu_{i+1}\},$$

and we use the symbol  $\mu$  both for the partition and its diagram. Most definitions and conventions used in this text are those of [5]. For example, the diagram of the partition  $(4, 2, 1)$  is geometrically represented as

2,0			
1,0	1,1		
0,0	0,1	0,2	0,3

and it consists of the lattice *cells*

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)\}.$$

Thus, the *coordinates*  $(r, c)$  of a cell are such that  $r + 1$  (resp.  $c + 1$ ) is the *row number* (resp. *column number*) for the cell position, counting from bottom up (resp. left to right).

Given a lattice diagram  $D = \{(r_1, c_1), (r_2, c_2), \dots, (r_n, c_n)\}$  we define the *lattice determinant*

$$\Delta_D(X; Y) := \det \|x_i^{r_j} y_i^{c_j}\|_{i,j=1}^n,$$

where  $X = x_1, x_2, \dots, x_n$  and  $Y = y_1, y_2, \dots, y_n$ . The polynomial  $\Delta_D(X; Y)$  is bihomogeneous of degree  $|r| = r_1 + \cdots + r_n$  in  $X$  and degree  $|c| = c_1 + \cdots + c_n$  in  $Y$ . To insure that this definition associates a unique polynomial to  $D$ , we order lattice cells in increasing lexicographic order.

We will need a few more definitions regarding partitions and diagrams. For an  $n$ -cell diagram  $D$ , a *tableau* of shape  $D$  is an injective map  $T : D \rightarrow \{1, 2, \dots, n\}$ . If  $T(r, c) = m$ , we say that  $h_T(m) := r$  is the *height* of  $m$  in  $T$ . We say that  $T$  is *row increasing* if  $T(i, j) < T(k, j)$  whenever  $i < k$  (when this has a meaning). Similarly we define *column increasing* tableaux, and a *standard* tableau is one that is both row and column increasing. We denote by  $\mathbb{S}_D$  the set of all standard tableau of shape  $D$ .

With these definitions out of the way, we come to our object of interest, namely the modules

$$M_D := \mathcal{L}_\partial[\Delta_D],$$

where  $D$  is some lattice diagram. The case where  $D = 1^n$  ( $1^n$  is the partition of  $n$  with all parts equal to 1) corresponds to the classical module of harmonic polynomials. This generalization was first considered by Garsia and Haiman [9], in the special case when  $D = \mu$  is the lattice diagram of a partition. Since then, several other cases have been studied (see [5] and [6]). For any  $n$ -cell lattice diagram  $D$ , the space  $M_D$

affords the structure of an  $\mathcal{S}_n$ -module, through the action of the symmetric group on polynomials consisting in permuting variables. More precisely, a permutation  $\sigma \in \mathcal{S}_n$  acts *diagonally* on a polynomial  $P(X; Y)$ , as follows

$$\sigma P(X; Y) = P(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_n}).$$

Under this action,  $\Delta_D = \Delta_D(X; Y)$  is clearly an alternant and the action commutes with partial derivatives, hence  $M_D$  is an invariant subspace of  $\mathbb{Q}[X, Y]$ .

Moreover, since  $\Delta_D$  is bihomogeneous,  $M_D$  affords the following natural bigrading. Denoting by  $\mathcal{H}_{r,s}[M_D]$  the subspace consisting of the bihomogeneous elements of degree  $r$  in  $X$  and degree  $s$  in  $Y$ , we have the direct sum decomposition

$$M_D = \bigoplus_{r=0}^{|p|} \bigoplus_{s=0}^{|q|} \mathcal{H}_{r,s}[M_D].$$

The *bigraded Hilbert series* of  $M_D$  is

$$F_D(q, t) = \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r q^s \dim \mathcal{H}_{r,s}[M_D],$$

and the bigraded character of  $M_D$  is encoded by the following symmetric function,

$$H_D(X; q, t) = \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r q^s \mathcal{F}(\text{ch} \mathcal{H}_{r,s}[M_D]),$$

where  $\text{ch} \mathcal{H}_{r,s}[M_D]$  denotes the character of  $\mathcal{H}_{r,s}[M_D]$  and  $\mathcal{F}$  is the Frobenius correspondence which maps the irreducible character  $\chi^\lambda$  to the Schur function  $s_\lambda$ . We say that this symmetric function is the *Bigraded Frobenius Characteristic* of  $M_D$ .

The  $n!$ -factorial conjecture of Garsia and Haiman states that for any partition diagram  $\mu$ ,  $H_\mu(X; q, t)$  is none other than a renormalized version of the Macdonald polynomial associated to  $\mu$ . This has recently been shown by Haiman [12] using an algebraic geometry approach. It develops that a very natural and combinatorial recursive approach to the  $n!$ -conjecture involves diagrams obtained by removing a single cell from a partition diagram. It is conjectured in [5] that, for such diagrams, the space  $M_D$  is a direct sum of left regular representations of  $\mathcal{S}_n$ . More precisely, if  $\mu$  is a partition of  $n+1$ , we denote by  $\mu/ij$  the lattice diagram obtained by removing one of its cell  $(i, j)$  from the diagram  $\mu$ . We refer to the cell  $(i, j)$  as the *hole* of  $\mu/ij$ . The conjecture in question states that the number of copies of the left regular representations in  $M_{\mu/ij}$  is equal to the cardinality of the  $(i, j)$ -shadow, that is the cardinality of  $\{(s, t) \in \mu : s \geq i, t \geq j\}$ .

This, and more, is all encoded in the following four term recurrence for the bigraded Frobenius characteristic  $H_{\mu/ij}$  of  $M_{\mu/ij}$ .

**Conjecture 1.1** ([5]). *For all  $(i, j) \in \mu$ , we have  $H_{\mu/ij} = C_{\mu/ij}$ .*

Where,  $C_{\mu/ij}$  is defined by the following the following “four term” recurrence

$$(1.2) \quad C_{\mu/ij} = \frac{t^\ell - q^{a+1}}{t^\ell - q^a} C_{\mu/i,j+1} + \frac{t^{\ell+1} - q^a}{t^\ell - q^a} C_{\mu/i+1,j} - \frac{t^{\ell+1} - q^{a+1}}{t^\ell - q^a} C_{\mu/i+1,j+1},$$

where  $\ell$  and  $a$  give the number of cells that are respectively north and east of  $(i, j)$  in  $\mu$ . As boundary conditions, we set  $C_{\mu/i,j+1}$ ,  $C_{\mu/i,j+1}$  or  $C_{\mu/i+1,j+1}$  equal to zero when the corresponding cells  $(i, j+1)$ ,  $(i+1, j)$  or  $(i+1, j+1)$  fall outside of  $\mu$ . Furthermore, if  $(i, j)$  is a corner of  $\mu$ , then  $\mu/ij$  is a partition diagram  $\nu$ , and we set

$$C_{\mu/ij} = H_\nu.$$

For any lattice diagram  $D$ , we consider  $M_D^0$  the  $Y$ -free component of  $M_D$ , this is to say that

$$M_D^0 = \bigoplus_{r=0}^{|p|} \mathcal{H}_{r,0}[M_D].$$

In this paper, we study the spaces  $M_{\mu/ij}^0$ , and show that the  $Y$ -free specialization of the conjecture 1.1 hold for these spaces. In particular this implies that

$$(1.3) \quad \dim M_{\mu/ij}^0 = \frac{n!}{\mu!} |\{ (r, c) \in \mu \mid i \leq r \leq \ell \}|,$$

where  $\ell$  is the largest integer for which the corresponding row of  $\mu$  has at least  $j$  cells. Moreover, we obtain a formula for  $H_{\mu/ij}^0$ , the graded Frobenius character of  $M_{\mu/ij}^0$ .

## 2. A BASIS FOR $M_\mu^0$

In preparation for our description of the modules  $M_{\mu/ij}^0$ , we need to recall and reformulate some results about the modules  $M_\mu^0$ . Although a recursive description of a basis for  $M_\mu^0$  is given in [7], and a direct description is given in [1], we give here a new description directly in term of standard tableaux of shape  $\mu$ . One can immediately link this description to Tanisaki’s construction [13] of the defining ideal of  $M_\mu^0$ . Moreover, we will see that it clearly generalizes the “Artin” basis for  $M_n = M_{1^n}^0$ :

$$\mathcal{B}_n := \{ \partial_X^{\mathbf{a}} \Delta_n(X) \mid \mathbf{a} = (a_1, a_2, \dots, a_n), \quad a_i < i \},$$

where we use the vectorial notation  $\partial_X^{\mathbf{a}} := \partial_{x_1}^{a_1} \partial_{x_2}^{a_2} \dots \partial_{x_n}^{a_n}$ .

First, let  $T$  be a tableau of shape  $D$  (any diagram), and define

$$Y_T := \prod_{(r,c) \in D} y_{T(r,c)}^c,$$

so that, for  $1 \leq k \leq n$ , the exponent of  $y_k$  is  $c$ , if  $(r, c)$  is the position of  $k$  in  $T$ . For example, if

$$T = \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 1 & 2 \\ \hline \end{array} \begin{array}{|c|c|c|c|c|} \hline & & 4 & 5 & 7 \\ \hline \end{array}$$

then  $Y_T = y_2 y_6 y_4^2 y_5^3 y_7^4$ . Clearly, the monomial  $Y_T$  encodes the columns in which the entries of  $T$  appear. More precisely, let the *column set* of a diagram  $T$  of shape  $D$ , be

$$\Gamma(T) := \{(k, \Gamma_k(T)) \mid 0 \leq k \leq \max_{(r,c) \in D} c\},$$

where for a given  $k$ , the set  $\Gamma_k(T)$  is the set of entries in column  $k$ , that is

$$\Gamma_k(T) := \{T(j, k) \mid (j, k) \in D\}.$$

Then  $Y_T = Y_R$ , if and only if  $T$  and  $R$  have same column sets (up to the addition of pairs of the form  $(k, \{\})$ ). Observe that  $T$  and  $R$  need not be of the same shape.

We will denote  $\partial Y_T$  the differential operator obtained by replacing each  $y_k$  in  $Y_T$  by  $\partial_{y_k}$ . With the same conventions, define

$$X_T := \prod_{(r,c) \in D} x_{T(r,c)}^r, \quad \text{and} \quad Z_T := X_T Y_T.$$

It is easy to see that  $T \mapsto Z_T$  establishes a bijection between injective tableaux of shape  $D$  and monomials of the form

$$x_{\sigma_1}^{r_1} y_{\sigma_1}^{c_1} x_{\sigma_2}^{r_2} y_{\sigma_2}^{c_2} \cdots x_{\sigma_n}^{r_n} y_{\sigma_n}^{c_n},$$

with  $\sigma$  in  $\mathcal{S}_n$  and  $D = \{(r_1, c_1), (r_2, c_2), \dots, (r_n, c_n)\}$  ordered lexicographically. If we set

$$\gamma_D := \prod_{(r,c) \in D} c!,$$

then we easily get the following lemma.

**Lemma 2.1.** *For two tableaux  $T$  and  $R$ , both of shape  $D$ , one has*

$$\partial Y_T Z_R = \gamma_D X_R$$

*if  $R$  can be obtained from  $T$  by a column fixing permutation of its entries. Otherwise*

$$\partial Y_T Z_R = 0.$$

Given a tableau  $T$  of shape  $D$ , we define the *Garnir polynomial*

$$\Delta_T(X) = \prod_{\gamma \in \Gamma(T)} \det \|x_m^{h_T(\ell)}\|_{m, \ell \in \gamma}.$$

Recall that  $h_T(\ell)$  is the height of  $\ell$  in  $T$ , this is to say that it is the first coordinate of the cell in which it appears in  $T$ . We have the following proposition.

**Proposition 2.2.** *For any tableau  $T$  of shape  $D'$ , we either have*

$$\partial Y_T \Delta_D(X, Y) = \pm \gamma_D \Delta_T(X),$$

*or  $\partial Y_T \Delta_D(X, Y)|_{Y=0} = 0$ .*

Clearly this last possibility can only occur if one of the column of  $D$  has a different number of cells than the corresponding column of  $D'$ .

*Proof.* We only give an outline of the proof, and restrict ourselves to the case when each column of  $D$  has the same number of cells as the corresponding column of  $D'$ . In that case,  $Y_T$  has the same total  $Y$ -degree as  $\Delta_D(X, Y)$ , so that  $\partial Y_T \Delta_D(X, Y)$  is a polynomial in the  $X$  variables only. Since the terms of  $\Delta_D(X, Y)$  are

$$\text{sign}(\sigma) x_{\sigma_1}^{r_1} y_{\sigma_1}^{c_1} x_{\sigma_2}^{r_2} y_{\sigma_2}^{c_2} \cdots x_{\sigma_n}^{r_n} y_{\sigma_n}^{c_n},$$

in view of lemma 2.1, the terms of  $\partial Y_T \Delta_D(X, Y)$  are forced to be of the form  $\partial Y_T Z_R$  for tableaux  $R$  that have the same column set as  $T$ . On the other hand, since  $\partial Y_T \Delta_\mu(X, Y)$  alternates in sign under the action of column fixing permutations, it has to be a multiple of  $\Delta_T(X)$ . Hence, both polynomials having the same degree, we must have the equality stated.  $\square$

We now associate to each entry  $j$ , of a standard tableau  $T$ , a non negative integer in the following manner. Let  $(r_j, c_j)$  be the position of  $j$  in  $T$ , and let  $k$  be the largest entry of  $T$ , such that  $c_k = c_j + 1$  and  $k < j$ . We set

$$\alpha(j) = \alpha_T(j) := r_j - r_k.$$

If there is no such  $k$ , set  $\alpha(j) := r_j + 1$ . For the example given below, the value of  $\alpha(k)$  appears in the cell of the right tableau corresponding to the position of  $k$  in the left tableau.

5						3					
4	8					2	2				
3	6					1	2				
1	2	7	9	10		1	1	1	1	1	

Clearly, if  $T$  is the unique standard tableau corresponding to a column:

$n$
$\vdots$
2
1

then  $\alpha(j) = j - 1$ .

As shown in [10, 7] the space  $\mathcal{L}_\partial[\Delta_T(X) : T \in \mathbb{S}_\mu]$  (the span of all partial derivative of Garnir polynomials for tableaux of shape  $\mu$ ) coincides with the space  $M_\mu^0$ . Using this characterization, we will now construct a basis for  $M_\mu^0$ . But first, let us introduce some further notation.

For  $\mu$  partition of  $n$ , let  $\pi(\mu)$  be the set of partitions of  $n - 1$  that can be obtained from  $\mu$  by removing one of its corner. For  $\nu \in \pi(\mu)$ , we denote  $\mu/\nu$  the corner by which  $\nu$  differs from  $\mu$ . Let us label  $\nu_1, \dots, \nu_k$ , the partitions in the set  $\pi(\mu)$ , (for definition, see the proof of theorem 2.3) following the increasing order of the column

number in which the corresponding corners, the  $\mu/\nu_i$ 's, appear. In other words, if  $(a_i, b_i)$ ,  $1 \leq i \leq k$ , are the respective coordinates of the corner cells  $\mu/\nu_i$ , then  $b_1 < b_2 < \dots < b_k$ . Any standard tableau  $T$  of shape  $\mu$  is such that  $n$  sits in a corner  $(a_j, b_j)$  of  $\mu$ . Moreover, the value of  $\alpha_T(n)$  depends only on the position of this corner (and on the shape  $\mu$ ), since all other entries of  $T$  are smaller. Denoting  $\alpha_j$  the value of  $\alpha_T(n)$ , if  $n$  appears in position  $(a_j, b_j)$  in  $T$ , it is clear that

$$(2.1) \quad \alpha_j = a_j - a_{j+1}.$$

**Theorem 2.3.** *For any partition  $\mu$  of  $n$ , the set of polynomials*

$$\mathcal{B}_\mu := \{ \partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_\mu, \mathbf{m} = (m_1, m_2, \dots, m_n) \text{ and } 0 \leq m_i < \alpha_T(i) \}$$

*is a basis of  $M_\mu^0$ .*

*Proof.* We first show recursively that  $\mathcal{B}_\mu$  is independent, assuming that the statement holds for partitions with at most  $n-1$  cells. As before, for  $\nu_j \in \pi(\mu)$ , let  $(a_j, b_j)$  be the corner  $\mu/\nu_j$ , with  $b_1 < b_2 < \dots < b_k$ , and define

$$\mathbf{B}_j := \{ X^{\mathbf{m}} \mid T \in \mathbb{S}_\mu, 0 \leq m_i < \alpha_T(i), T(a_j, b_j) = n \}.$$

In view of 2.1, for  $X^{\mathbf{m}} \in \mathbf{B}_j$ , the dominant monomial of  $\partial_X^{\mathbf{m}} \Delta_T(X)$  (in reverse lexicographic order) is of the form

$$x_n^{a_j - m_n} X^{\mathbf{p}} \quad (\text{where } p_n = 0),$$

with  $0 \leq m_n < a_j - a_{j+1}$ . For  $k$  fixed with  $a_{j+1} < k \leq a_j$ , our induction hypothesis gives that the set

$$\mathcal{B}_{j,k} := \{ \partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_\mu, 0 \leq m_i < \alpha_T(i), T(a_j, b_j) = n, m_n = a_j - k \},$$

is independent, since (in reverse lexicographic order) we have the following expansion of  $\partial_X^{\mathbf{m}} \Delta_T(X)$

$$\partial_X^{\mathbf{m}} \Delta_T(X) = x_n^k \partial_X^{\mathbf{p}} \Delta_{T'}(X) + \underbrace{\dots}_{\text{lower terms}}$$

where  $T'$  is the restriction of  $T$  to  $\nu_j$ . Clearly, the sets  $\mathcal{B}_{j,k}$  are mutually independent, so

$$(2.2) \quad \mathcal{B}_\mu = \bigsqcup_{j,k} \mathcal{B}_{j,k}$$

is independent.

We will now show that the number of elements of  $\mathcal{B}_\mu$  is

$$(2.3) \quad |\mathcal{B}_\mu| = \frac{n!}{\mu!},$$

using a recursive argument, assuming that the statement holds for  $\nu_j \in \pi(\mu)$ . By induction, it is clear that

$$|\mathcal{B}_{j,k}| = \frac{(n-1)!}{\nu_j!}$$

so that (in view of 2.2)

$$|\mathbf{B}_\mu| = \sum_{i=1}^k \alpha_j \frac{(n-1)!}{\nu_j!}.$$

The result follows from the easy observation that

$$n = \sum_{j=1}^k \alpha_j (a_j + 1),$$

since

$$a_j + 1 = \frac{\mu!}{\nu_j!}$$

is the length of the row of  $\mu$  in which sits the corner  $(a_j, b_j)$ . A "geometric" argument, that can be found in [11], shows that the dimension of  $M_\mu^0$  is at most  $\frac{n!}{\mu!}$ . Thus  $\mathcal{B}_\mu$  is a basis.  $\square$

It is shown in [11] that the graded Frobenius character of the  $M_\mu^0$ 's are none other than the Hall-Littlewood symmetric functions.

### 3. A BASIS FOR $M_{\mu/ij}^0$

The central result of this paper is the following description of a basis for  $M_{\mu/ij}^0$ , with  $\mu$  any fixed partition of  $n+1$ , and  $(i, j)$  any given cell of  $\mu$ . The proof that it is a generator set is postponed until the next section.

For a standard tableau  $T$ , let  $\mathbf{B}_T$  simply denote the set

$$\mathbf{B}_T := \{ X^{\mathbf{m}} \mid 0 \leq m_s \leq \alpha_T(s) \}.$$

For  $\nu_\ell$  a partition of  $n$  obtained from  $\mu$  by removing the corner cell  $(a_\ell, b_\ell)$ , the basis of  $M_\ell := M_{\nu_\ell}$ , described in the previous section, is

$$\mathcal{B}_\ell = \{ \partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_{\nu_\ell}, X^{\mathbf{m}} \in \mathbf{B}_T \}.$$

If  $T$  is a standard tableau of shape  $\nu_\ell$ , and  $0 \leq u \leq a_\ell$  an integer, we denote  $T \uparrow_{uv}$  the tableau of shape  $\mu/uv$  (with  $v = b_\ell$ ), such that

$$T \uparrow_{uv}(r, c) = \begin{cases} T(r, c) & \text{if } c \neq v \text{ or } r < u, \\ T(r-1, c) & \text{if } c = v \text{ and } r > u. \end{cases}$$

Since  $(u, v)$  is not in  $\mu/uv$ ,  $T \uparrow_{uv}$  need not be defined at  $(u, v)$ . In other words, the tableau  $T \uparrow_{uv}$  is obtained from  $T$  by "sliding" upward by 1 the cells in column  $v$  that are on or above row  $u$ . For  $u$  and  $v$  as above, we set

$$(3.1) \quad \mathcal{A}_{uv} := \{ \partial_X^{\mathbf{m}} \Delta_{T \uparrow_{u,v}}(X) \mid X^{\mathbf{m}} \in \mathbf{B}_T, T \in \mathbb{S}_{\nu_\ell} \}.$$



Observe that  $\mathcal{A}_{uv}$  implicitly depends on the choice of corner  $(a_\ell, b_\ell)$  of  $\mu$ , since  $v$  is equal to  $b_\ell$ . Moreover,  $\mathcal{A}_{a_\ell, b_\ell}$  is the basis of  $M_{\nu_\ell}^0$  described in theorem 2.3, thus  $\mathcal{A}_{a_\ell, b_\ell}$  is independent.

Let  $(a_1, b_1), \dots, (a_m, b_m)$  (with  $b_1 < \dots < b_m$ ) be the set of corners of  $\mu$  that are in the “shadow” of  $(i, j)$ . This is to say that  $i \leq a_\ell$  and  $j \leq b_\ell$ , for all  $1 \leq \ell \leq m$ . Once again, we denote  $\alpha_\ell$  the value of  $\alpha_T(n+1)$  for any standard tableau of shape  $\mu$  having  $n+1$  in position  $(a_\ell, b_\ell)$ . Defining

$$(3.2) \quad \mathcal{B}_{\mu/ij} := \bigcup_{\ell=1}^m \bigcup_{u=i}^{\min(i+\alpha_\ell-1, a_\ell)} \mathcal{A}_{u, b_\ell},$$

we have

**Theorem 3.1.** *For  $\mu$  a partition of  $n+1$  and  $(i, j) \in \mu$ ,  $\mathcal{B}_{\mu/ij}$  is a basis of  $M_{\mu/ij}^0$ .*

*Proof.* In the remainder of this section, we will prove that 3.2 is an independent set, using a downward recursive argument, and that the number of elements of  $\mathcal{B}_{\mu/ij}$  is

$$(3.3) \quad d_{\mu/ij} := \frac{n!}{\mu!} \sum_{\substack{i' > i \\ \mu_{i'} > j}} \mu_{i'}.$$

To complete the proof of the theorem, we will show in section 4 that the dimension of  $M_{\mu/ij}^0$  is at most equal to  $d_{\mu/ij}$ , so that  $M_{\mu/ij}^0$  has to coincide with the span of  $\mathcal{B}_{\mu/ij}$ .  $\square$

The cardinality of  $\mathcal{A}_{u, b_\ell}$  is clearly

$$\frac{n!}{\mu!} \mu_{a_\ell+1-u+i} = \frac{n!}{\mu!} \mu_{a_\ell+1}$$

since this is the same as  $n!/\nu_\ell!$ . If 3.2 is a disjoint union (which would follow from it being independent) and since every  $i'$  indexing the summation in 3.3 occur exactly once in  $\{a_\ell + 1 - u + i \mid 1 \leq \ell \leq m, i \leq u \leq \min(i + \alpha_\ell - 1, a_\ell)\}$ , then we must have

$$(3.4) \quad |\mathcal{B}_{\mu/ij}| = d_{\mu/ij}.$$

Let

$$D_X := \partial_{x_1} + \partial_{x_2} + \dots + \partial_{x_n}.$$

We recall the following special case of proposition (I.2) of [5].

**Proposition 3.2.** *If  $(i+1, j)$  is in  $\mu$ , then*

$$D_X \Delta_{\mu/i, j}(X, Y) =_{\text{cte}} \Delta_{\mu/i+1, j}(X, Y).$$

*Otherwise  $D_X \Delta_{\mu/ij}(X, Y) = 0$ , and this corresponds to the case where  $(i, j)$  is on the top border of  $\mu$ . The symbol “ $=_{\text{cte}}$ ” stands for equality up to a non zero constant.*

It is easy to adapt the proof of this fact to show that, for  $T$  a standard tableau of shape  $\nu_\ell$  (with  $\mu/\nu_\ell = (a_\ell, b_\ell)$ ) and  $0 \leq u \leq a_\ell$ , we have

$$(3.5) \quad D_X \Delta_{T \uparrow_{u,v}}(X) =_{\text{cte}} \begin{cases} \Delta_{T \uparrow_{u+1,v}}(X) & \text{if } u < a_\ell, \\ 0 & \text{if } u = a_\ell, \end{cases}$$

where  $v = b_\ell$ . It follows from definition 3.1 that

**Lemma 3.3.** *Using the same convention as above, we have*

$$(3.6) \quad D_X \mathcal{A}_{u,b_\ell} =_{\text{cte}} \begin{cases} \mathcal{A}_{u+1,b_\ell} & \text{if } u < a_\ell, \\ \{0\} & \text{if } u = a_\ell. \end{cases}$$

Since these two sets have the same cardinality we deduce, from the linear independence of  $\mathcal{A}_{a_\ell,b_\ell}$ , that each  $\mathcal{A}_{u,b_\ell}$  is independent. Applying  $D_X$  in definition 3.2 we readily check that

$$(3.7) \quad \mathcal{B}_{\mu/i+1,j} = D_X \mathcal{B}_{\mu/i,j}.$$

But we know that  $D_X \mathcal{A}_{a_m,b_m} = \{0\}$ , and it is clear that  $\mathcal{A}_{a_m,b_m}$  is a subset of  $\mathcal{B}_{\mu/ij}$ . A dimension count, together with the recursive assumption, forces  $\mathcal{B}_{\mu/i+1,j}$  to be independent, since

$$d_{\mu/i+1,j} + |\mathcal{A}_{a_m,b_m}| = \frac{n!}{\mu!} \sum_{\substack{i' > i+1 \\ \mu_{i'} > j}} \mu_{i'} + \frac{n!}{\mu!} \mu_{i+1} = d_{\mu/ij}.$$

This ends the proof of theorem 3.1.

#### 4. UPPER BOUND FOR THE DIMENSION OF $M_{\mu/ij}^0$

We now give an upper bound for the dimension of  $M_{\mu/ij}^0$ . Given a polynomial  $P(X, Y)$ , we denote by  $P(\partial)$  the operator obtained from  $P$  by replacing all the variables  $x_i$  and  $y_j$  by  $\partial_{x_i}$  and  $\partial_{y_j}$ , respectively. If  $P(X, Y) = M$  is a single monomial  $M$  we will write  $P(\partial) = \partial_M$ .

**Theorem 4.1.** *For  $\mu$  a partition of  $n+1$ ,*

$$\dim M_{\mu/i,j}^0 \leq \frac{n!}{\mu!} \sum_{\substack{i' > i \\ \mu_{i'} > j}} \mu_{i'}.$$

*Proof.* In [5], the bigraded  $\mathcal{S}_n$ -modules  $M_{\mu/i,j}$  and  $\mathcal{L}_\partial[\partial_{x_{n+1}}^i \partial_{y_{n+1}}^j \Delta_\mu(X; Y)]$  are shown to be equivalent, hence their  $Y$ -free components

$$M_{\mu/i,j}^0 \quad \text{and} \quad \mathcal{L}_\partial[\partial_{x_{n+1}}^i \partial_{y_{n+1}}^j \Delta_\mu(X; Y)]^0,$$

are equivalent. For any injective tableau  $T$  of shape  $\mu$ , with  $c_{n+1} \geq j$ , we have  $Y_T = y_{n+1}^j Y^{\mathbf{a}}$  and

$$\pm \gamma_\mu \Delta_T(X) = \partial Y_T \Delta_\mu(X, Y).$$

If  $r_{n+1} < i$  then  $\partial_{x_{n+1}}^i \Delta_T(X) = 0$ , so that  $\mathcal{L}_\partial[\partial_{x_{n+1}}^i \partial_{y_{n+1}}^j \Delta_\mu(X; Y)]^0$  is equal to

$$(4.1) \quad \mathcal{L}_\partial[\partial_{x_{n+1}}^i \Delta_T(X) \mid T: \mu \rightarrow \{1, 2, \dots, n+1\}, c_{n+1} \geq j, r_{n+1} \geq i].$$

For  $\zeta_1, \zeta_2, \dots, \zeta_{\ell(\mu)}$  and  $\omega_1, \omega_2, \dots, \omega_{\ell(\mu')}$  two families of pairwise distinct scalars, we construct a set of points  $[\rho_\mu]_{i,j}$  in  $\mathbb{C}^{2(n+1)}$  as follow. For every injective tableau  $T$  of shape  $\mu$ , we define the point  $\rho_T = (\zeta_{r_1}, \zeta_{r_2}, \dots, \zeta_{r_{n+1}}, \omega_{c_1}, \omega_{c_2}, \dots, \omega_{c_{n+1}})$  in  $\mathbb{C}^{2(n+1)}$ , and set

$$[\rho_\mu]_{i,j} = \{\rho_T \mid T: \mu \rightarrow \{1, 2, \dots, n+1\}, c_{n+1} \geq j, r_{n+1} \geq i\}.$$

That is  $\rho_T \in [\rho_\mu]_{i,j}$  when  $n+1$  lies in the shadow of  $(i, j)$  in  $T$ . Note that  $\rho_T \in [\rho_\mu] = [\rho_\mu]_{0,0}$  contain  $n!$  points in correspondance with every injective tableau of shape  $\mu$ .

We denote by  $[\rho_\mu]_{i,j}^0 = \pi([\rho_\mu]_{i,j})$ , where  $\pi$  is the projection on  $\mathbb{C}^{n+1}$  that keeps only the first  $n+1$  entries. We see that the set of tableaux with  $n+1$  entries strictly increasing in rows and where  $n+1$  lies in a row  $i'$  such that  $i' = r_{n+1} + 1 > i$  and  $\mu_{i'} = c_{n+1} + 1 > j$  give all the points of  $[\rho_\mu]_{i,j}^0$  exactly once. One then easily verifies that the cardinality of  $[\rho_\mu]_{i,j}^0$  is precisely

$$d_{\mu/ij} = \frac{n!}{\mu!} \sum_{\substack{i' > i \\ \mu_{i'} > j}} \mu_{i'}.$$

Following [5, Section 4] we associate to this set  $J_{[\rho_\mu]_{i,j}^0}$ , its annihilator ideal and define  $H_{[\rho_\mu]_{i,j}^0} = (J_{[\rho_\mu]_{i,j}^0})^\perp$ . The dimension of  $H_{[\rho_\mu]_{i,j}^0}$  is then  $d_{\mu/ij}$  as well.

Given a polynomial  $P$ , let  $h(P)$  denotes its homogeneous component of highest degree. For any polynomial  $P$  in  $J_{[\rho_\mu]_{i,j}^0}$ , let

$$Q(X, Y) = P(X) \prod_{i'=1}^i (x_{n+1} - \zeta_{i'}) \prod_{j'=1}^j (y_{n+1} - \omega_{j'})$$

For any  $\rho_T \in [\rho_\mu]$ , the two products in the definition of  $Q$  vanish at  $\rho_T$  unless  $n+1$  lies in the shadow of  $(i, j)$  in  $T$ . But if this is the case then  $P(X)$  vanishes at  $\pi(\rho_T)$ . This shows that  $Q(X, Y)$  is in  $J_{[\rho_\mu]}$ , the annihilator ideal of  $[\rho_\mu]_{0,0} = [\rho_\mu]$ . Hence  $h(Q) = h(P)x_{n+1}^i y_{n+1}^j$  is in  $\text{gr}(J_{[\rho_\mu]})$ , its graded version and  $h(Q)(\partial) \Delta_\mu(X, Y) = 0$ . For any injective tableau  $T$  of shape  $\mu$  such that  $c_{n+1} \geq j$  and  $r_{n+1} \geq i$  we have

$Y_T = y_{n+1}^j Y^{\mathbf{a}}$  and

$$\begin{aligned} h(P)(\partial) \partial_{x_{n+1}}^i \Delta_T(X) &= \pm \gamma_\mu^{-1} h(P)(\partial) \partial_{x_{n+1}}^i \partial Y_T \Delta_\mu(X, Y) \\ &= \pm \gamma_\mu^{-1} h(P)(\partial) \partial_{x_{n+1}}^i \partial_{y_{n+1}}^j \partial_Y^{\mathbf{a}} \Delta_\mu(X, Y) \\ &= \pm \gamma_\mu^{-1} \partial_Y^{\mathbf{a}} h(Q)(\partial) \Delta_\mu(X, Y) = 0 \end{aligned}$$

Thus  $h(P)$  is in  $I_{\partial_{x_{n+1}}^i \Delta_T(X)}$ . We obtain this way that  $\text{gr} J_{[\rho_\mu]_{i,j}^0}$  is a subset of  $I_{\partial_{x_{n+1}}^i \Delta_T(X)}$  for any  $T$  with the prescribed conditions. The space in 4.1 is thus contained in  $H_{[\rho_\mu]_{i,j}^0}$ , which proves the theorem.  $\square$

## 5. FOUR TERM RECURRENCE

Specializing conjecture 1.1 to its  $Y$ -free component, corresponds to setting  $q = 0$  in the four term recurrence 1.2. We now show that this specialization of conjecture 1.1 holds, by giving an explicit interpretation of the resulting recurrence in term of the basis we have constructed for  $M_{\mu/ij}^0$ .

**Theorem 5.1.** *If  $H_{\mu/ij}^0$  denotes the graded Frobenius characteristic of  $M_{\mu/ij}^0$  then:*

- if  $a = 0$  and  $\ell > 0$ ,  $H_{\mu/ij}^0 = \frac{t^{\ell+1} - 1}{t^\ell - 1} H_{\mu/i,j+1}^0$ ;
- if  $a > 0$ ,  $H_{\mu/ij}^0 = H_{\mu/i,j+1}^0 + t H_{\mu/i+1,j}^0 - t H_{\mu/i+1,j+1}^0$ .
- if  $a = 0$  and  $\ell = 0$ ,  $H_{\mu/ij}^0$  is the graded Frobenius characteristic of  $M_\nu^0$ , where  $\nu$  is the partition  $\mu/ij$ .

Here (as before)  $\ell$  and  $a$  give the number of cells that are respectively north and east of  $(i, j)$  in  $\mu$ . If any of the cells  $(i+1, j)$ ,  $(i, j+1)$  or  $(i+1, j+1)$  falls out of  $\mu$ , then the corresponding term is considered to be 0.

*Proof.* Each of these assertions can be shown using the basis we have constructed. The third one is just a direct observation. The first one corresponds to a case for which there is just one corner  $(a_m, b_m)$  in the shadow of  $(i, j)$ , with  $b_m = j$ , and then 3.2 can be written as

$$\mathcal{B}_{\mu/ij} := \bigcup_{u=0}^{\ell} \mathcal{A}_{i+u,j},$$

Since, as long as  $(k+1, j)$  is in  $\mu$ ,  $D_X$  is an isomorphism of representations between the homogeneous  $\mathcal{S}_n$ -modules  $\mathcal{A}_{k,j}$  and  $\mathcal{A}_{k+1,j}$  that lowers the degree by 1, we must have

$$\mathcal{F}_t(\mathcal{A}_{k,j}) = t \mathcal{F}_t(\mathcal{A}_{k+1,j})$$

where  $\mathcal{F}_t$  stands for the graded Frobenius characteristic. We deduce that, in the first case,

$$\mathcal{B}_{\mu/ij} = (1 + t + \dots + t^\ell) H_\nu^0$$

with  $\mu/\nu = (a_m, b_m)$ . This is clearly equivalent to the statement of the first case.

For the second case there are a few subcases, all similarly dealt with, the most interesting one being when  $j = b_1$  and  $m > 1$  for which the basis can clearly be broken down as

$$\mathcal{B}_{\mu/ij} = \mathcal{B}_{\mu/i, j+1} \uplus \bigcup_{u=i}^{i+\alpha_1-1} \mathcal{A}_{u, b_1}$$

and we only need to show that the graded Frobenius characteristic of the linear span of

$$\bigcup_{u=i}^{i+\alpha_1-1} \mathcal{A}_{u, b_1}$$

is given by

$$(5.1) \quad t(H_{\mu/i+1, j}^0 - H_{\mu/i+1, j+1}^0).$$

Now we clearly have  $\mathcal{B}_{\mu/i+1, j+1} \subset \mathcal{B}_{\mu/i+1, j}$ , with  $D_X \bigcup_{u=i}^{i+\alpha_1-1} \mathcal{A}_{u, b_1}$  being the complement of  $\mathcal{B}_{\mu/i+1, j+1}$  in  $\mathcal{B}_{\mu/i+1, j}$ . Under the hypothesis of this subcase, the graded Frobenius characteristic of the span of  $\bigcup_{u=i}^{i+\alpha_1-1} \mathcal{A}_{u, b_1}$  is thus given by 5.1. All other subcases are simple to show.  $\square$

## 6. REMARKS

**Remark 6.1.** In [7] (proposition 2.2), N. Bergeron and Garsia show that the spaces  $M_\mu^0$  are nested into each other according to their partition indexing. That is

$$\mu \preceq \lambda \quad \implies \quad M_\mu^0 \subseteq M_\lambda^0$$

where “ $\preceq$ ” denotes the dominance order. Moreover they show that

$$M_\mu^0 \cap M_\lambda^0 = M_{\mu \wedge \lambda}^0.$$

Using our basis, it is easy to show that both these results extend to the situation studied in this paper. Namely,

**Proposition 6.2.** *For two partition  $\mu$  and  $\lambda$  of  $n+1$ , we have*

$$\mu \preceq \lambda \quad \implies \quad M_{\mu/ij}^0 \subseteq M_{\lambda/ij}^0$$

and

$$M_{\mu/ij}^0 \cap M_{\lambda/ij}^0 = M_{\mu \wedge \lambda / ij}^0,$$

whenever  $(i, j)$  appears in both  $\mu$  and  $\lambda$ .

**Remark 6.3.** For  $\mu$  a partition of  $n$  (denoted  $\mu \vdash n$ ), Macdonald has given an explicit description of the coefficients appearing in the Pieri formula for the  $H_\mu$ :

$$h_k^\perp H_\mu(X; q, t) = \sum_{\substack{\nu \vdash n-k \\ \nu \subseteq \rho}} c_{\mu\nu}^k(q, t) H_\nu(X; q, t)$$

where  $h_k^\perp$  is the operator dual to multiplication by  $h_k$  (complete homogeneous) with respect to the usual scalar product on symmetric functions for which the Schur functions are orthonormal. These coefficients  $c_{\mu\nu}^k(q, t)$  are rational functions in  $q$  and  $t$ . Now, let  $\rho$  be the partition of  $m$  corresponding to the shadow of  $(i, j)$  in  $\mu$ , with  $m$  equal to the number of cells in this shadow. F. Bergeron has conjectured in [4] that the following symmetric function

$$(6.1) \quad \sum_{\substack{\nu \vdash m-k \\ \nu \subseteq \rho}} c_{\mu\nu}^k(q, t) H_{\mu-\rho+\nu}(X; q, t),$$

where  $\mu - \rho + \nu$  stands for the partition obtained from  $\mu$  by replacing  $\rho$  (the shadow of  $(i, j)$ ) by  $\nu$ , is the bigraded Frobenius characteristic of the module  $M_{\mu/ij}^k$  obtained as the union of all modules  $M_D$ , for  $D$  ranging in the set of diagrams obtained from  $\mu$  by removing  $k$  cells in the shadow of  $(i, j)$ . This would imply that the dimension of  $M_{\mu/ij}^k$  be equal to  $\binom{m}{k} (n-k)!$ . J.-C. Aval, in [2], has shown that this value is an upper bound, and has generalized the construction of this paper to obtain an explicit basis for the  $Y$ -free component of  $M_{\mu/ij}^k$ . One can show that the graded Frobenius characteristic of the resulting space is the symmetric function obtained by taking the limit as  $q \rightarrow 0$  of 6.1.

**Remark 6.4.** One can explicitly characterize the defining ideal of the space  $M_{\mu/ij}^0$ . This will be the subject of a forthcoming paper [3].

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(Jean-Christophe Aval) LABORATOIRE A2X, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE

(François Bergeron) DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DU QUÉBEC À MONTRÉAL, MONTRÉAL, QUÉBEC, H3C 3P8, CANADA.

(Nantel Bergeron) DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, TORONTO, ONTARIO M3J 1P3, CANADA

*E-mail address*, Jean-Christophe Aval: [aval@math.u-bordeaux.fr](mailto:aval@math.u-bordeaux.fr)

*E-mail address*, François Bergeron: [bergeron.francois@uqam.ca](mailto:bergeron.francois@uqam.ca)

*E-mail address*, Nantel Bergeron: [bergeron@mathstat.yorku.ca](mailto:bergeron@mathstat.yorku.ca)

*URL*, François Bergeron: <http://bergeron.math.uqam.ca>

*URL*, Nantel Bergeron: <http://www.math.yorku.ca/bergeron>